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The Diffraction of Electric Waves around a Finite, Perfectly Conducting Cone

Part II: The Field Singularities

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In this Part it is found that the electromagnetic field set up by a finite, perfectly conducting cone in the presence of an oscillating electric dipole becomes singular at the tip of the cone and also at its rim, and the nature of these singularities is determined.

I. INTRODUCTION

In Part I the electromagnetic field set up by a finite, perfectly conducting cone, in the presence of an electric dipole situated upon its axis of symmetry and pointing along its axis, was determined.

Letting a be the distance of the dipole from the cone vertex, taking spherical polar coordinates (r, θ, ϕ) based on the tip and axis of the cone (the latter being defined by $0 < r < l$, $\theta = \alpha$ and $0 < \theta < \alpha$, $r = l$) and taking all field vectors proportional to the harmonic time factor exp $(i\omega t)$, it was shown that the field is given in terms of a scalar wave function Π by

$$\mathbf{E} = -\frac{1}{ra}\left\{\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\Pi}{\partial\mu};\ -\frac{\partial}{\partial r}r\left(\frac{\partial\Pi}{\partial\theta}\right);\ 0\right\}$$

$$\mathbf{H} = \left\{0;\ 0;\ -\frac{ik}{a}\frac{\partial\Pi}{\partial\theta}\right\} \tag{1}$$

where $\mu = \cos\theta$, $k = 2\pi/\lambda = \omega/c$ (c = velocity of light in vacuo).

In "region I" $(0 < r < l, \alpha < \theta < \pi)$

$$\Pi = r^{-1}\sum_{0}^{\infty} A_j\psi_{\beta_j}(kr)P_{\beta_j}(-\mu) \tag{2}$$

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where the β_j are the positive zeros of $P_x(-\cos\alpha)$, $\psi_n(x) = (\frac{1}{2}\pi x)^{1/2} J_{n+1/2}(x)$ and,

in "region II" $(r > l)$,

$$\Pi = (e^{-ikR}/R) + r^{-1} \sum_0^\infty B_n \zeta_n(kr) P_n(\mu) \tag{3}$$

where

$$\begin{aligned} R^2 &= r^2 + a^2 + 2\,\text{ar}\cos\theta \\ \zeta_n(x) &= (\tfrac{1}{2}\pi x)^{\frac{1}{2}} H^{(2)}_{n+\frac{1}{2}}(x) \end{aligned} \tag{4}$$

and the coefficients A_j, B_n in the above are given by

$$A_j \psi_{\beta_j} F(\beta_j) = \sum_{n=0}^\infty \left\{ -\frac{i}{ka}(2n+1)(-1)^n \bar{\zeta}_n \psi_n + B_n \zeta_n \right\} F_n(\beta_j) \tag{5}$$

and the B_n by

$$\beta_n = \frac{i}{ka}(-1)^n (2n+1) \frac{C_n + \bar{\zeta}_n \psi_n'}{\zeta_n'} \tag{6}$$

where the C_n satisfy

$$\begin{aligned} C_n \operatorname{cosec}^2\alpha = {} & -2P_n(-\mu_0) \sum_{j=0,m=0}^\infty \sum^\infty \frac{(m+\frac{1}{2})^3}{\beta_j + \frac{1}{2}} \frac{i\bar{\zeta}_m + C_m \zeta_m}{\zeta_m'} \frac{\psi'_j}{\psi_{\beta_j}} \\ & \times \frac{P_m(-\mu_0)}{(n-\beta_j)(n+\beta_j+1)(m-\beta_j)(m+\beta_j+1)} \\ & \times \frac{P'_{\beta_j}(-\mu_0)}{[\partial P_s(-\mu_0)/\partial s]_{(\beta_j)}} + \frac{iK}{2} \frac{1}{n(n+1)} P_n'(-\mu_0) \end{aligned} \tag{7}$$

where

$$\begin{aligned} F_n(\beta_j) &= \frac{2}{\pi} \frac{\sin\pi\beta_j}{(n-\beta_j)(n+\beta_j+1)} \frac{P_n(\mu_0)}{P_{\beta_j}(\mu_0)} \\ F(\beta_j) &= \frac{2}{\pi} \frac{\sin\pi\beta_j}{2\beta_j+1} \frac{[\partial P_s(-\mu_0)/\partial s]}{P_{\beta_j}(\mu_0)} \qquad (s = \beta_j) \\ \mu_0 &= \cos\alpha \end{aligned} \tag{7A}$$

and in (5)–(7) $\bar{\zeta}_n$ is written for $\zeta_n(ka)$, ψ_n for ψ_n for $\psi_n(kl)$ etc., and K is a constant which is determined by substituting the solution of (7) $\{C_n(K)$, say,$\}$ into the condition

$$\sum_0^\infty (2m+1) \frac{i\bar{\zeta}_m + C_m \zeta_m}{\zeta_m'} P_m(-\mu_0) = 0 \tag{8}$$

which the C's were shown to satisfy. This obviously produces an equation for K.

It will be appear in the present Part that the field outside the cone becomes singular at the cone tip and also at its rim, and we proceed now to investigate the nature of these singularites.

II. The Field Near the Cone Tip

By (1) and (11) we have, for sufficiently small r

$$\mathbf{E} = A_0 \frac{(k/2)^{\beta_0+1}}{\Gamma(\beta_0 + \frac{3}{2})} (\beta_0 + 1) \sqrt{\pi}\{\beta_0 P_{\beta_0}(-\mu); -\sin\theta P'_{\beta_0}(-\mu); 0\} r^{\beta_0-1} \tag{9}$$

The electric field near the tip therefore tends to infinity like $r^{-(1-\beta_0)}$ and the singularity disappears if we make $\alpha \geqslant \pi/2$, since β_j steadily increases with α, being equal to j at $\alpha = 0$, $2j + 1$ at $\alpha = \pi/2$, and tending to infinity as $\alpha \to \pi - 0$. If α is made to approach π the approach of β_j towards $+\infty$ shows that, in the case of a conical hole bored to the center of a conducting sphere, very little of the field will penetrate to the bottom of the hole.

Regarding the magnetic field $\mathbf{H}$, it is easy to verify that this tends to zero as the tip is approached.

It is further seen that, as might be expected, the tip singularity is of the same nature as that for the semi-infinite cone.

III. The Field Near the Rim

By (1) and (2) for points in region I and by (1) and (3) for points in region II, the field components at points outside the cone are expressed by certain infinite series. Since the terms of these series are finite and continuous through any finite region outside the cone, *and* at points on its surface, it follows that any singularity in the field will be expressed by these series becoming divergent, as happens, for example, at the dipole position $r = a$, $\theta = \pi$. We shall, therefore, examine carefully the convergence properties of these series, with special reference to an approach to the cone rim.

A. The Approach from Region II

It will be most convenient to begin by considering approaches from this region, as the series in (3) then gives the part of the field which becomes large, and the behavior of the B_n for large n is known via (6) and via the important result (proved in the Appendix)

$$C_{s-\frac{1}{2}} \sim (iK \sin^2\alpha) s^{-2} P'_{s-\frac{1}{2}}(-\cos\alpha) \tag{10}$$

where the constant K is the value of $a\{\partial(r\Pi)/\partial r\}$ upon the base surface (see Eq. (13), Part I).

The result (10) holds for all sufficiently large s provided that s is not in an immediate[1] neighborhood of large zeros of $P'_{s-1/2}(-\mu_0)$; otherwise (10) does not dominate the small error terms.

Now, for large n,

$$\zeta_n \psi_n' \sim (\text{const})(l/a)^n \tag{11}$$

and so, by (10), for positive integral large n,

$$(\zeta_n \psi_n')/C_n \sim O\{n^{3/2}(l/a)^n\} \tag{12}$$

and so, since $(l/a) < 1$, (6) gives

$$\begin{aligned}\beta_n &\sim \frac{i}{ka}(-1)^n(2n+1)(C_n/\zeta_n') \\ &\sim -\frac{2K\sin^2\alpha}{ka}(-1)^n \frac{1}{n\zeta_n'} P_n'(-\mu_0)\end{aligned} \tag{13}$$

by (10).

In considering the neighborhood of the rim, only the series part of (3) $(\hat{\Pi})$ is significant, as the dipole function $\Pi_0(=\exp(-ikR)/R)$ and all its derivatives, are finite at the rim. By (13) the large order terms in the series part of (3) (the disturbance function Π), are given by

$$\Pi = \sum_0^\infty \hat{\Pi}_n$$

$$\begin{aligned}\hat{\Pi}_n &\sim \frac{2K\sin^2\alpha}{kar} n^{-1} \frac{\zeta_n(kr)}{\zeta_n'(kl)} P_n'(\mu_0)P_n(\mu) \\ &\sim -\frac{4Kl}{\pi a r}\left(\frac{\sin\alpha}{\sin\theta}\right)^{\frac{1}{2}} n^{-2}\left(\frac{l}{r}\right)^n \sin\left[(n+\tfrac{1}{2})\alpha - \frac{\pi}{4}\right]\cos\left[(n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right]\end{aligned} \tag{14}$$

where we have used

$$\begin{aligned}\zeta_n(x) &\sim \frac{i}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{-n}\Gamma(n+\tfrac{1}{2}) \\ P_n(\cos\theta) &\sim \left(\frac{2}{n\pi\sin\theta}\right)^{\frac{1}{2}}\cos\left\{(n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\}\end{aligned} \tag{15}$$

[1] i.e., its distance from these points $\gg |s|^{-1}$.

Therefore, by (1),

$$H_\phi \sim -\sum \frac{4iKkl}{\pi a^2 r}\left(\frac{\sin\alpha}{\sin\theta}\right)^{\frac{1}{2}} n^{-1}\left(\frac{l}{r}\right)^n \sin\left[(n+\tfrac{1}{2})\alpha - \frac{\pi}{4}\right]\sin\left[(n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right]$$

$$= -\sum \frac{2iKkl}{\pi a^2 r}\left(\frac{\sin\alpha}{\sin\theta}\right)^{\frac{1}{2}} n^{-1}\left(\frac{l}{r}\right)^n \{\cos[(n+\tfrac{1}{2})(\theta-\alpha)] - \sin[(n+\tfrac{1}{2})(\theta+\alpha)]\} \tag{16}$$

Hence, as the rim is approached from region II,

$$\begin{aligned} H_\phi &\sim -\frac{2iKk}{\pi a^2}\,\mathrm{Re}\left(\sum_0^\infty n^{-1}\left(\frac{l}{r}\right)^n e^{(n+\frac{1}{2})(\theta-\alpha)i}\right) \\ &\sim \quad \frac{2iKk}{\pi a^2}\,\mathrm{Re}\, e^{\frac{1}{2}i(\theta-\alpha)} \ln\left(1 - \frac{l}{r}e^{i(\theta-\alpha)}\right) \\ &\sim -\frac{2iKk}{\pi a^2}\ln\frac{l}{\epsilon} \end{aligned} \tag{17}$$

where ϵ is the shortest distance of a general field point from the rim and so is given by

$$\epsilon = [l^2 + r^2 - 2lr\cos(\theta-\alpha)]^{1/2} \tag{18}$$

We shall next investigate the behavior of the electric field $\mathbf{E}$ as the rim is approached from region II. By (1) we have the general relations

$$\begin{aligned} E_r &= \frac{1}{ikr}\,\frac{\partial}{\sin\theta\,\partial\theta}(H_\phi\sin\theta) \\ E_\theta &= -\frac{1}{ikr}\,\frac{\partial}{\partial r}(rH_\phi) \end{aligned} \tag{19}$$

(which are also apparent from the relation $ik\mathbf{E} = \mathrm{curl}\,\mathbf{H}$). Therefore, as the rim is approached

$$\begin{aligned} E_r &\sim \frac{1}{ikl}\left(\frac{\partial H_\phi}{\partial\theta}\right) \\ &\sim \frac{2K}{\pi a^2}(\epsilon^{-1}\sin\gamma) \end{aligned} \tag{20}$$

$$\begin{aligned} E_\theta &\sim -\frac{1}{ik}\left(\frac{\partial H_\phi}{\partial r}\right) \\ &\sim -\frac{2K}{\pi a^2}(\epsilon^{-1}\cos\gamma) \end{aligned} \tag{21}$$

on differentiating (17) and using (18), where ϵ is, as just mentioned, the least distance of the field point from the rim and γ is the angle made by this least distance with the outward directed cone generator through its meeting point with the rim. The angle γ is measured in the sense of θ increasing—see Fig. 1.

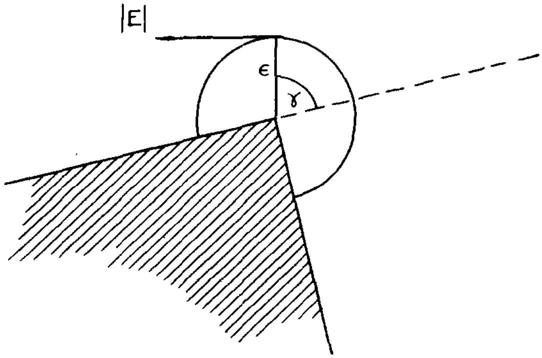

FIG. 1. $|E| \sim (2K/\pi a^2)\epsilon^{-1}$, $|H| \sim (2Kk/\pi a^2) \ln (l/\epsilon)$, is perpendicular to a meridian plane (plane of the paper) and 90° out of time-phase with $|E|$.

B. The Approach from Region I

We have now to consider approaches to the rim from region I. Since the field in this region is given by (2) we need for this purpose a knowledge of the behavior of the A_j therein, for large j. It proves to be surprisingly difficult to attain this, as will be seen from what follows.

By (5) and (6)

$$A_j \psi_{\beta_j} F(\beta_j) = \frac{i}{ka} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{i\bar{\zeta}_n + C_n \zeta_n}{\zeta_n'} F_n(\beta_j) \tag{21}$$

i. e.,

$$A_j = \frac{i}{ka} \frac{2\beta_j + 1}{\psi_{\beta_j}} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{i\bar{\zeta}_n + C_n \zeta_n}{\zeta_n'} \times \frac{P_n(\mu_0)}{(n - \beta_j)(n + \beta_j + 1)[\partial P_s(-\mu_0)/\partial s]_{(\beta_j)}} \tag{22}$$

For large j we have

$$\beta_j + \tfrac{1}{2} \sim \frac{(j + \frac{3}{4})\pi}{\pi - \alpha} \tag{23}$$

$$[\partial P_s(-\mu_0)/\partial s]_{(\beta_j)} \sim -\left(\frac{2}{\pi \sin \alpha}\right)^{\frac{1}{2}} \frac{\pi - \alpha}{\sqrt{\beta_j}} (-1)^j \tag{24}$$

Thus, for large j

$$(-1)^j A_j \beta_j^{\frac{1}{2}} \psi_{\beta_j} \sim -\frac{2i}{ka}\left(\frac{\pi \sin\alpha}{2}\right)^{\frac{1}{2}} \frac{1}{\pi-\alpha}$$

$$\times \sum_{n=0}^{\infty} (2n+1) \frac{i\bar{\zeta}_n + C_n \zeta_n}{\zeta_n'} \frac{\beta_j^2}{(n+\frac{1}{2})^2 - (\beta_j + \frac{1}{2})^2} P_n(-\mu_0) \tag{25}$$

Now

$$\lim_{\beta_j \to \infty} \left(\frac{\beta_j^2}{(n+\frac{1}{2})^2 - (\beta_j+\frac{1}{2})^2}\right) = 1 \tag{26}$$

for each fixed n, and the use of (10) for large order C_n shows that the above series converges uniformly for large β_j including "$\beta_j = +\infty$" (that is, the series is uniformly convergent in any interval $0 \leqslant Z$, where $Z = 1/\beta_j$).

Accordingly, letting $j \to \infty$ in (25), and using the result (8), we find that

$$\lim_{j\to\infty} (-1)^j A_j \beta_j^{\frac{1}{2}} \psi_{\beta_j} = 0 \tag{27}$$

In view of this result, it is natural to now consider $(-1)^j A_j \beta_j^{5/2} \psi_{\beta_j}$, after using the identity

$$\frac{1}{(n+\frac{1}{2})^2 - (\beta_j+\frac{1}{2})^2} = -\frac{1}{(\beta_j+\frac{1}{2})^2} + \frac{(n+\frac{1}{2})^2}{[(n+\frac{1}{2})^2 - (\beta_j+\frac{1}{2})^2](\beta_j+\frac{1}{2})^2} \tag{28}$$

to transform the series in (25), while at the same time again making use of (8): this process readily gives

$$(-1)^j A_j \beta_j^{5/2} \psi_{\beta_j} \sim -\frac{4i}{ka}\left(\frac{\pi\sin\alpha}{2}\right)^{\frac{1}{2}} \frac{1}{\pi-\alpha} \sum_{n=0}^{\infty} (n+\tfrac{1}{2})^3$$

$$\times \frac{i\bar{\zeta}_n + C_n\zeta_n}{\zeta_n'} \frac{\beta_j^2}{(n+\frac{1}{2})^2 - (\beta_j+\frac{1}{2})^2} P_n(-\mu_0) \tag{29}$$

but we cannot now deduce the behavior of $(-1)^j A_j \beta_j^{5/2} \psi_{\beta_j}$ as $j \to \infty$ by letting $j \to \infty$ in the equation and using the limit (26) because

$$\sum (n+\tfrac{1}{2})^3 \frac{i\bar{\zeta}_n + C_n\zeta_n}{\zeta_n'} P_n(-\mu_0) \tag{30}$$

diverges owing to the divergence of the part involving the C_n (as may easily be seen on using formula (10) for the large order C_n). There is no difficulty about the part of the series (29) involving $\bar{\zeta}_n/\zeta'_n$ as $\bar{\zeta}_n/\zeta_n' \sim -kln^{-1}(l/a)^n$ for large order n, and $l < a$. Hence the requirements (referred to above) concerning the uniformity of convergence of this part, as $\beta_j \to \infty$, are satisfied.

Accordingly, as $j \to \infty$ in (29) it becomes

$$(-1)^j A_j \beta_j^{5/2} \psi_{\beta_j} \sim O(1) - \frac{4i}{ka}\left(\frac{\pi \sin \alpha}{2}\right)^{\frac{1}{2}} \frac{\beta_j^2}{\pi - \alpha} \sum_{n=0}^{\infty} (n + \tfrac{1}{2})^3$$
$$\times \frac{C_n \zeta_n}{\zeta_n'} \frac{P_n(-\mu_0)}{(n + \frac{1}{2})^2 - (\beta_j + \frac{1}{2})^2} \tag{31}$$

and we have now to approximate directly to the *sum function* of the series on the right of (31), when β_j is large. Now

$$\zeta_n/\zeta_n' \sim -kl/(n + \tfrac{1}{2}) \tag{32}$$

for $n \gg x$ and $n \gg 1$, and the formula (10) for large order C_n is also true for sufficiently large n (independent of β_j). Hence by (31) and (10)

$$\sum_{n=0}^{\infty} (n + \tfrac{1}{2})^3 \frac{C_n \zeta_n}{\zeta_n'} \frac{P_n(-\mu_0)}{(n + \frac{1}{2})^2 - (\beta_j + \frac{1}{2})^2}$$

$$= O(\beta_j^{-2}) - ikKl \sin^2 \alpha \sum_{n=0}^{\infty} \frac{P_n'(\mu_0) P_n(\mu_0)}{(n + \frac{1}{2})^2 - (\beta_j + \frac{1}{2})^2}$$

$$= O(\beta_j^{-2}) - \frac{ikKl}{2} \sin^2 \alpha \int_{-\infty}^{\infty} \frac{P'_{iy-\frac{1}{2}}(\mu_0) P_{iy-\frac{1}{2}}(-\mu_0)}{y^2 + (\beta_j + \frac{1}{2})^2} \operatorname{sech} \pi y \, dy \tag{33}$$

on transforming the sum by contour integration. This has been effected by taking $\sec s\pi \cdot P'_{s-1/2}(\mu_0) P_{s-1/2}(-\mu_0)/[s^2 - (\beta_j + \frac{1}{2})^2]$ around a very large semicircle bounded on the left by the imaginary axis. Use of the asymptotic formula (15) for the large order Legendre functions shows that the contribution of the curved part of the contour tends to zero as its radius tends to infinity because the integrand behaves thereon like $| s |^{-2}$.

Using the asymptotic formula for the Legendre functions again in the above infinite integral will obviously cause an error in the value of the integral of order β_j^{-2}; therefore (33) is

$$\sum_{n=0}^{\infty} (n + \tfrac{1}{2})^3 \frac{C_n \zeta_n}{\zeta_n'} \frac{P_n(-\mu_0)}{(n + \frac{1}{2})^2 - (\beta_j + \frac{1}{2})^2}$$
$$= O(\beta_j^{-2}) + \frac{ikKl}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{y^2 + (\beta_j + \frac{1}{2})^2}$$
$$= \frac{ikKl}{2} \beta_j^{-1} + O(\beta_j^{-2}) \tag{34}$$

Hence

$$(-1)^j A_j \beta_j^{\frac{3}{2}} \psi_{\beta_j} \sim \frac{Kl}{a(\pi - \alpha)} (2\pi \sin \alpha)^{\frac{1}{2}} \tag{35}$$

as $j \to \infty$

We can now determine the field behavior of an approach to the rim from region I; from (1) and (2), and the use of (35) to substitute for the large order A_j in (2), and, on remembering that $\psi_{\beta_j}(kr)/\psi_{\beta_j}(kl) \sim (r/l)^{\beta_j+1/2}$, we have[2]

$$\begin{aligned}
H_\phi &= -\frac{ik}{a}\frac{\partial \Pi}{\partial \theta} \\
&\sim -\frac{ik}{la^2}\frac{Kl}{\pi-\alpha}(2\pi \sin \alpha)^{\frac{1}{2}} \sum \beta_j^{-1}\left(\frac{2}{\pi \sin \alpha}\right)^{\frac{1}{2}} (r/l)^{\beta_j} \cos\left[(\beta_j + \tfrac{1}{2})(\theta - \alpha)\right] \\
&\sim -\frac{2ikK}{\pi a^2} \sum j^{-1} (r/l)^{\pi j/(\pi-\alpha)} \cos\left[(j + \tfrac{3}{4})\frac{\pi}{\pi - \alpha}(\theta - \alpha)\right] \\
&\sim \frac{2ikK}{\pi a^2} \operatorname{Re}\left\{ e^{\frac{3}{4}(\pi/\pi-\alpha)(\theta-\alpha)i} \ln\left[1 - \left(\frac{r}{l}\right)^{\pi/(\pi-\alpha)} \exp\left(\frac{\pi}{\pi-\alpha}(\theta - \alpha)i\right)\right]\right\} \\
&\sim -\frac{2ikK}{\pi a^2} \ln \frac{l}{\epsilon}
\end{aligned} \tag{36}$$

where ϵ is the distance from the rim.

This is exactly the same law as was found in (17) for the approach to the rim from region II and it now follows from the general relation $ik\mathbf{E} = \operatorname{curl} \mathbf{H}$ (exactly as in (19)) that the law of approach for $\mathbf{E}$ will likewise be the same Hence we have now proved that, for *general* approaches to the rim. $\mathbf{E}$ becomes singular like $\epsilon^{-1}(|E| \sim (2K/\pi a^2)\epsilon^{-1})$ and the electric field lines in the rim vicinity become small circles centered on the rim whose planes are perpendicular to the rim direction at their centers. The magnetic field lines, are, of course, always circles centered on the axis of symmetry whose planes are perpendicular to that axis. See Fig. 1.

IV. Discussion of Results

The above results show that theories which assume that the part of the external field "contributed by secondary radiation from the rim singularity" is equivalent to a cumulative effect of diffraction fields from supposedly equivalent small 90° wedges disposed around the rim, are incorrect, because they would give the wrong type of singularity there. For the fields near the edge of a semi-infinite conducting wedge behave like $\epsilon^{-1/2}$.

[2] $P_{\beta_j}(-\mu) \sim [2/(\pi \sin \alpha)^{1/2}(-1)^j \sin(\beta_j + \frac{1}{2})(\theta - \alpha)$.

The above finding, that the rim neighborhood field is not "wedge-like," comes as no surprise to this writer.

For, looking first at the problem generally, there always remains the fundamental difference between the two diffraction problems, namely, that one is a finite body problem whilst the other is a semi-infinite body problem. Consequently, electro-magnetic energy can always circulate freely and indefinitely often *around* the finite cone, whereas this is clearly impossible for the semi-infinite wedge for which its only method of entry into the shadow zone is by Sommerfeld type diffraction across the straight edge. It therefore seems (to this writer) reasonable to expect that the magnitude of the field in the neighborhood of the rim of a finite cone should be of larger order than that obtaining for fields near edges of a semi-infinite conducting wedge. This is certainly the case for **E**, and also for **H**, for the part $-(2ikK/\pi a^2)\ln(l/\epsilon)$ of **H** should be neglected for the purposes of this argument because, as will be seen immediately, it does not contribute to the energy flow away from the rim.

Consider now this matter of secondary radiation from the rim edge. In the case of the semi-infinite wedge diffraction problem, both **E** and **H** are $O(\epsilon^{-1/2})$ and this is of the right order of magnitude to produce radiation from the edge. With regard now to the cone, in order to assess the radiation possibilities for the rim (since we are now dealing with a finite body problem) we would have to, by the requirements of the Poynting vector radiation theorem, enclose the rim with a *closed* surface.

This, however, cannot be done without the surface passing through the cone interior within which all the field expressions of this paper do not apply. But, within the cone, we may assume that $\mathbf{E} \equiv 0$, for the interior of a perfect conductor cannot support electric field, and this property also implies $\mathbf{E} \equiv 0$ through any hollow interior parts. With this assumption we can, therefore, take the Poynting energy flux integral $c/4\pi \iint (\mathbf{E} \wedge \mathbf{H}) \cdot d\mathbf{S}$ over the whole of the thin anchor-ring surface surrounding the rim, defined by the property that the shortest distance of any point on it from the rim equals ϵ. Since by (17), (20), and (21) the large (surface) components of **E** and **H** upon this tubular surface are 90° *out of time-phase with each other* outside the cone, and since $\mathbf{E} \equiv 0$ *inside* the cone, the contribution of the dominant part of $\mathbf{E} \wedge \mathbf{H}$ (as obtained from (17), (20), (21)) to the time-mean of the Poynting integral through an integral number of time cycles, is zero.

Nevertheless, the results (17), (20), (21) show that there *is* radiation of energy from the cone rim. For the next highest order term in the product $\mathbf{E} \wedge \mathbf{H}$ is obtained by multiplying the dominant part of $\mathbf{E}(O(\epsilon^{-1})$, as given by (20), (21)) by the finite part of **H** (i.e., $H_\phi + (2iKk/\pi a^2)\ln(l/\epsilon)$). Since it appears from the series analysis of (16) that (by uniform convergence) the finite part of **H** is continuous up to, and including, the rim position, the next highest order term of $\mathbf{E} \wedge \mathbf{H}$ gives, by (20) and (21), a contribution to the Poynting integral over the thin ring-shaped surface surrounding the rim, which is not in general zero. For one would require the finite part of $\mathbf{H}/(iK)$ at the rim of the real or zero to make the time-mean of the contribution in question to the Poynting integral over the rim-enclosing surface above to vanish. Hence, there will, in general, be secondary radiation from the rim.

We can now obtain a very interesting result about the secondary radiation from the rim. Let us enclose the whole cone by a closed surface Σ whose parts approximate

(i) to the part of the tubular surface surrounding the rim which lies *outside* the cone,

(ii) to the slant surface excluding the rim and tip neighborhoods, and,

(iii) to the portion of a small sphere surrounding the tip which lies outside the cone.

Since the tangential component of $\mathbf{E}$ is zero at the cone's surface and since $\mathbf{E}$ is continuous at all points outside it (except at the dipole position) (ii) obviously gives no contribution to the Poynting radiation surface integral over Σ. Next, since the tip singularity makes $\mathbf{E} = 0\ (r^{\beta_0-1})$ and $\mathbf{H} = O(r^{\beta_0})$ (c.f. Section II) the contribution of (iii) to the said Poynting surface integral over Σ is also zero.

The secondary radiation sent back from the cone therefore must come entirely from the rim.

In conclusion, perhaps it would provide a little further insight into conditions obtaining near the cone rim to remark that, even apart from the above exact analysis, the hypothesis of wedge-like rim vicinity fields seems obviously wrong because of inconsistency with (1). For such a hypothesis certainly entails the existence of *some* field singularity at the rim but by (1) it then follows that $\mathbf{E}$ will have a higher order rim singularity than will $\mathbf{H}$, because its components are given by higher order derivatives of the generating function Π than is the case for $\mathbf{H}$. Thus the hypothesis of wedge-like rim vicinity fields is contradicted, for this requires the same order singularity for $\mathbf{E}$ and $\mathbf{H}$ $(O(\epsilon^{-1/2}))$.

Appendix I. Behavior of $C_{s-\frac{1}{2}}$ for Large s

For general s, $C_{s-\frac{1}{2}}$ is defined by writing $n = s - \frac{1}{2}$ in the right of (7). Inversion of the order of summation of the double sum appearing in (7) has been justified in Part I (preceding this paper); consequently, (7) is really a set of linear equations of the form

$$C_n = p_n + \sum_{m=0}^{\infty} q_{nm} C_m \tag{A.1}$$

To make any progress, therefore, we have to know the behavior of the q_{nm} for large (complex) $n(= s - \frac{1}{2})$ and so we have to begin by approximating to the known j sum in (7).

The sum with regard to β_j in (7) is

$$\sum_{j=0}^{\infty} \frac{1}{(\beta_j + \frac{1}{2})[s^2 - (\beta_j + \frac{1}{2})^2][(m + \frac{1}{2})^2 - (\beta_j + \frac{1}{2})^2]} \frac{\psi'_{\beta_j}}{\psi_{\beta_j}} \frac{P'_{\beta_j}(-\mu_0)}{[\partial P_s(-\mu_0)/\partial s]_{(\beta_j)}}$$

$$= -\frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{1}{z(s^2 - z^2)[(m + \frac{1}{2})^2 - z^2]} \frac{\psi'_{z-\frac{1}{2}}}{\psi_{z-\frac{1}{2}}} \frac{P'_{z-\frac{1}{2}}(-\mu_0)}{P_{z-\frac{1}{2}}(-\mu_0)} dz$$

$$+ \frac{1}{2} \frac{1}{s^2 - (m + \frac{1}{2})^2}$$

$$\times \left[\frac{1}{(m + \frac{1}{2})^2} \frac{\psi'_m}{\psi_m} \frac{P'_m(-\mu_0)}{P_m(-\mu_0)} - \frac{1}{s^2} \frac{\psi'_{s-\frac{1}{2}}}{\psi_{s-\frac{1}{2}}} \frac{P'_{s-\frac{1}{2}}(-\mu_0)}{P_{s-\frac{1}{2}}(-\mu_0)} \right]$$

$$- \sum_{r=1}^{N} \frac{1}{\alpha_r(s^2 - \alpha_r^2)[(m + \frac{1}{2})^2 - \alpha_r^2]} \frac{\psi'_{\alpha_r - \frac{1}{2}}}{[\partial \psi_{s-\frac{1}{2}}/\partial s]_{(\alpha_r)}} \frac{P'_{\alpha_r - \frac{1}{2}}(-\mu_0)}{P_{\alpha_r - \frac{1}{2}}(-\mu_0)} \tag{A.2}$$

on transforming the sum by contour integration, where $0 < \kappa < \beta_0$ and the α_r are the zeros of $\psi_{z-\frac{1}{2}}$ qua function of z. These zeros lie on the line segment $0 < z < x$ $(x = kl)$ and their number N does not exceed $(2x/\pi) + 2$.

The above transformation is justifiable because the integrand in the above behaves like $|z|^{-3}$ upon a large semicircle bounded on the left by the line $\mathrm{Re}(z) = \kappa$.

To gain the form of the above series for large s, we need to approximate to the above line integral. This is easily done, since for large $|z|$ $(\gg x)$

$$\psi'_{z-\frac{1}{2}}/\psi_{z-\frac{1}{2}} \sim (z + \tfrac{1}{2})/x \tag{A.3}$$

and for z, $|\mathrm{Im}(z)|$ larger than about 3,

$$P'_{z-\frac{1}{2}}(-\mu_0)/P_{z-\frac{1}{2}}(-\mu_0) \sim \pm iz/\sin\alpha \tag{A.4}$$

according as $I(z) \gtrless 0$ and since the conditions under which these approximations hold are independent of s, we have

$$\int_{\kappa - i\infty}^{\kappa + i\infty} \frac{1}{z(s^2 - z^2)[(m + \frac{1}{2})^2 - z^2]} \frac{\psi'_{z-\frac{1}{2}}}{\psi_{z-\frac{1}{2}}} \frac{P'_{z-n}(-\mu_0)}{P_{z-n}(-\mu_0)} dz$$

$$\sim \frac{2i}{x \sin\alpha} \int_0^{i\infty} \frac{z}{(s^2 - z^2)[(m + \frac{1}{2})^2 - z^2]} dz + O(s^{-2})$$

$$\sim -\frac{2i}{x \sin\alpha} \frac{1}{s^2 - (m + \frac{1}{2})^2} \ln \frac{s}{m + \frac{1}{2}} \tag{A.5}$$

Before gaining the required approximation to the $C_{s-\frac{1}{2}}$ it will be convenient to estimate the order of magnitude, for large s, of the contribution to it of the part of the double sum in (7) *not* involving the C_m : this part is

$$-2P_{s-\frac{1}{2}}(-\mu_0)\sum_{m=0}^{\infty}(m+\tfrac{1}{2})^3\frac{\bar{\zeta}_m}{\zeta_m'}F_m(s)P_m(-\mu_0) \tag{A.6}$$

where $F_m(s)$ denotes the β_j sum of (A.2).

Now, by (A.2) and (A.5)

$$\lim_{|s|\to\infty} s^2(\ln s)^{-1}F_m(s) = -2i/(x \sin\alpha) \tag{A.7}$$

for each fixed m.

Also, if s tends to infinity through the sequence $N + \frac{1}{2}$ (N integral) (A.5) and (A.2) show that (A.7) is still true for each fixed m, but that

$$\begin{aligned} F_m(s) &\sim O(s^{-2}) \\ &\to 0 \end{aligned} \tag{A.8}$$

for those $m \sim s(= N + \frac{1}{2})$.

If s tends to infinity through $\beta_N + \frac{1}{2}$

$$\begin{aligned} P_{s-\frac{1}{2}}(-\mu_0)F_m(s) &= \frac{1}{s^2}\frac{\psi'_{s-\frac{1}{2}}}{\psi_{s-\frac{1}{2}}}P'_{s-k}(-\mu_0)\frac{1}{s^2-(m+\frac{1}{2})^2} \\ &= O(s^{-5/2}) \end{aligned} \tag{A.9}$$

except for those $m \sim s(= \beta_N + \frac{1}{2})$, when this expression is $0(s^{-\frac{3}{2}})$.

Therefore, certainly,

$$| F_m(s)(\ln s)^{-1}s^2 | < 3/(x \sin\alpha) \tag{A.10}$$

for all $| s |$ sufficiently large (but not of the form $\beta_N + \frac{1}{2}$).

Therefore, since, for sufficiently large m

$$\begin{aligned} | (m+\tfrac{1}{2})^3(\bar{\zeta}_m/\zeta_m')P_m(-\mu_0) | &\sim m^2(l/a)^m(kl) \\ &< (\text{const})\, m^2(l/a)^m \end{aligned} \tag{A.11}$$

it follows by (A.10) and the M-test of Weierstrass that the series

$$\sum_0^{\infty} s^2(\ln s)^{-1}F_m(s)(m+\tfrac{1}{2})^3(\bar{\zeta}_m/\zeta_m')P_m(-\mu_0)$$

is uniformly convergent for all such large s, and hence by (A.7) and (A.8) that series (A.6) is at most of order $s^{-2}(\ln s)P_{s-\frac{1}{2}}(-\mu_0)$, except for s arbitrarily close to zeros of $P_{s-\frac{1}{2}}(-\mu_0)$.

By a similar argument using (A.9) it can be shown that, when s tends to infinity through the sequence $\beta_N + \frac{1}{2}$, then (A.6) is $O(s^{-5/2})$.

The Approximation to $C_{s-\frac{1}{2}}$ for Large s

We shall show in this section that, for large enough s, then

$$C_{s-\frac{1}{2}} \sim As^{-1}P_{s-\frac{1}{2}}(-\mu_0) + Bs^{-2}P'_{s-\frac{1}{2}}(-\mu_0) \tag{A.12}$$

where A and B are constants and we proceed to determine these constants (and at the same time verify the truth of hypothesis (A.12) by substituting it into the text equation (7) for the $C_{s-\frac{1}{2}}$ ($n = s - \frac{1}{2}$). Hypothesis (A.12) is suggested by the form of (7).

In doing this, we may neglect the part of the double sum in (7) involving the factor ζ_m/ζ_m' in the summand, because this has just been shown to be of smaller order than hypothesis (A.12), for sufficiently large s.

Taking account, therefore, only of the part of the double sum of text equation (7) involving factor C_m in the summand, and using (A.2) to substitute for the sum on the β_j, then this equation becomes, with $n = s - \frac{1}{2}$ and s large:

$$
\begin{aligned}
&s^{-1}(A \operatorname{cosec}^2 \alpha)P_{s-\frac{1}{2}}(-\mu_0) + s^{-2}\left(B \operatorname{cosec}^2 \alpha - \frac{iK}{2}\right) P_{s-\frac{1}{2}}(-\mu_0) \\
&\quad = \frac{1}{\pi i} P_{s-\frac{1}{2}}(-\mu_0) \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{1}{z(s^2-z^2)} \frac{\psi'_{z-\frac{1}{2}}}{\psi_{z-\frac{1}{2}}} \frac{P'_{z-\frac{1}{2}}(-\mu_0)}{P_{z-\frac{1}{2}}(-\mu_0)} \\
&\qquad \times \sum_0^\infty \frac{(m+\frac{1}{2})^3}{(m+\frac{1}{2})^2 - z^2} \frac{\zeta_m}{\zeta_m'} C_m P_m(-\mu_0) \\
&\qquad - P_s(-\mu_0) \sum_0^\infty \frac{m+\frac{1}{2}}{(m+\frac{1}{2})^2 - s^2} \frac{\psi_m'}{\psi_m} \frac{\zeta_m}{\zeta_m'} C_m P_m'(-\mu_0) \\
&\qquad + \frac{1}{s^2} \frac{\psi'_{s-\frac{1}{2}}}{\psi_{s-\frac{1}{2}}} P'_{s-\frac{1}{2}}(-\mu_0) \sum_0^\infty \frac{(m+\frac{1}{2})^3}{(m+\frac{1}{2})^2 - s^2} \frac{\zeta_m}{\zeta_m'} C_m P_m(-\mu_0) \\
&\qquad + 2P_{s-\frac{1}{2}}(-\mu_0) \sum_{r=1}^N \frac{1}{\alpha_r(s^2-\alpha_r^2)} \frac{\psi'_{\alpha_r-\frac{1}{2}}}{[\partial\psi_{s-\frac{1}{2}}/\partial s]_{(\alpha_r)}} \frac{P'_{\alpha_r-\frac{1}{2}}(-\mu_0)}{P_{\alpha_r-\frac{1}{2}}(-\mu_0)} \\
&\qquad \times \sum_0^\infty \frac{(m+\frac{1}{2})^3}{(m+\frac{1}{2})^2 - \alpha_r^2} \frac{\zeta_m}{\zeta_m'} C_m P_m(-\mu_0)
\end{aligned} \tag{A.13}
$$

and we have now to approximate to the right hand side of this equation using the fact that s is large and using also the hypothesis (A.12) for the *large order* C_m. In the summation on m on the right of (A.13) we may use

$$\begin{aligned} \zeta_m/\zeta_m' &\sim -x/(m+\tfrac{1}{2}) \\ \psi_m'/\psi_m &\sim (m+\tfrac{1}{2})/x \end{aligned} \tag{A.14}$$

and the approximation hypothesis

$$C_m \sim A(m+\tfrac{1}{2})^{-1}P_m(-\mu_0) + \beta(m+\tfrac{1}{2})^{-2}P_m'(-\mu_0) \tag{A.15}$$

for m large enough $> M_0$, say, and M_0 is independent of s. Therefore, using (A.14) and (A.15) throughout the series on the right of (A.13) leads (by application of (A.5) to approximate the integral in the case $m \leqslant M_0$) to error terms of at most $0\{s^{-2}(\ln s)P_{s-\frac{1}{2}}(-\mu_0)\}$, and these are negligible compared with the order of (A.12).

Also, the terms involving $\sum_{r=1}^{N}$ on the right of (A.13) are obviously $0\{s^{-2}P_{s-\frac{1}{2}}(-\mu_0\}$ and so are also negligible compared with (A.12).

Thus, (A.13) becomes, for large s

$$\begin{aligned} &(A\operatorname{cosec}^2\alpha)s^{-1}P_{s-\frac{1}{2}}(-\mu_0) + \left(B\operatorname{cosec}^2\alpha - \frac{iK}{2}\right)s^{-2}P'_{s-\frac{1}{2}}(-\mu_0) \\ &= -\frac{x}{\pi i}P_{s-\frac{1}{2}}(-\mu_0)\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{1}{z(s^2-z^2)}\frac{\psi'_{z-\frac{1}{2}}}{\psi_{z-\frac{1}{2}}}\frac{P'_{z-\frac{1}{2}}(-\mu_0)}{P_{z-\frac{1}{2}}(-\mu_0)} \\ &\quad\times\sum_0^\infty\frac{P_m(\mu_0)}{(m+\frac{1}{2})^2-z^2}[A(m+\tfrac{1}{2})P_m(-\mu_0)+BP_m'(-\mu_0)]\,dz \\ &\quad + P_{s-\frac{1}{2}}(-\mu_0)\sum_0^\infty\frac{P_m'(-\mu_0)}{(m+\frac{1}{2})[(m+\frac{1}{2})^2-s^2]} \\ &\quad\times[A(m+\tfrac{1}{2})P_m(-\mu_0)+BP_m'(-\mu_0)] \\ &\quad - \frac{1}{s}P'_{s-\frac{1}{2}}(-\mu_0)\sum_0^\infty\frac{P_m(-\mu_0)}{(m+\frac{1}{2})^2-s^2}[A(m+\tfrac{1}{2})P_m(-\mu_0)+BP_m'(-\mu_0)] \end{aligned} \tag{A.16}$$

To simplify further the righthand side when s is large we need the following results, easily proved by contour integration around a contour D consisting of a large semicircle bounded on the left by the imaginary axis of t.

They hold for all s making $\mathrm{Re}(s) > 0$.

$$\sum_{m=0}^{\infty} \frac{(m+\frac{1}{2})P_m{}^2(-\mu_0)}{(m+\frac{1}{2})^2 - s^2} - \frac{\pi}{2} P_{s-\frac{1}{2}}(-\mu_0)P_{s-\frac{1}{2}}(\mu_0) \sec s\pi$$

$$= -\frac{1}{2i}\int_D \frac{tP_{t-\frac{1}{2}}(-\mu_0)P_{t-\frac{1}{2}}(\mu_0)}{t^2 - s^2} \sec \pi t \, dt$$

$$= 0 \tag{A.17}$$

$$\sum_{m=0}^{\infty} \frac{P_m{}'(-\mu_0)P_m(-\mu_0)}{(m+\frac{1}{2})^2 - s^2} - \frac{\pi}{2s} P_{s-\frac{1}{2}}(-\mu_0)P_{s-\frac{1}{2}}(\mu_0) \sec s\pi$$

$$= -\frac{1}{2i}\int_D \frac{P'_{t-\frac{1}{2}}(-\mu_0)P_{t-\frac{1}{2}}(\mu_0)}{t^2 - s^2} \sec \pi t \, dt$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} \frac{P'_{iy-\frac{1}{2}}(-\mu_0)P_{iy-\frac{1}{2}}(\mu_0)}{y^2 + s^2} \operatorname{sech} \pi y \, dy$$

$$\sim \left(\frac{1}{2\sin^2\alpha}\right) s^{-1} \tag{A.18}$$

for large s, since for $|y| \gg 1$ (say > 3)

$$P'_{iy-}(-\mu_0)P_{iy-\frac{1}{2}}(\mu_0) \operatorname{sech} \pi y \sim -\frac{1}{\pi \sin^2 \alpha} \tag{A.19}$$

$$\sum_{0}^{\infty} \frac{[P_m{}'(-\mu_0)]^2}{(m+\frac{1}{2})[(m+\frac{1}{2})^2 - s^2]} + \frac{\pi}{2s^2} P'_{s-\frac{1}{2}}(-\mu_0)P'_{s-\frac{1}{2}}(\mu_0) \sec s\pi$$

$$= \frac{1}{2i}\int_{\dot{D}} \frac{P'_{t-\frac{1}{2}}(-\mu_0)P'_{t-\frac{1}{2}}(\mu_0)}{t(t^2 - s^2)} \sec \pi t \, dt$$

$$= \frac{\pi}{2} s^{-2} P'_{-\frac{1}{2}}(-\mu_0)P'_{-\frac{1}{2}}(\mu_0) \tag{A.20}$$

where $\dot{D}$ means that the contour has been indented at 0.

It is easy to verify from the asymptotic formula (see, e.g., text equation (15)) for large order Legendre functions, that all the integrands in the above behave like $|t|^{-2}$ upon the semicircular part of the contour D, and so the contribution of the curved part of it to the contour integrals tends to zero as the radius is made to tend to infinity.

Using the above formulas (A.17)–(A.20) and the identity

$$P_{s-\frac{1}{2}}(\mu)P'_{s-\frac{1}{2}}(-\mu) + P'_{s-\frac{1}{2}}(\mu)P_{s-\frac{1}{2}}(-\mu) = -(2/\pi)\operatorname{cosec}^2\alpha \cos s\pi \tag{A.21}$$

(established by the writer in Appendix II) we find at once that the terms on the right of (A.16) *not* involving the line integral are equal to

$$-\frac{1}{2}AP_{s-\frac{1}{2}}(-\mu_0)\int_{-\infty}^{\infty}\frac{P'_{iy-\frac{1}{2}}(-\mu_0)P_{iy-\frac{1}{2}}(\mu_0)}{y^2+s^2}\operatorname{sech}\pi y\,dy$$

$$+\frac{\pi}{2}Bs^{-2}P'_{-\frac{1}{2}}(-\mu_0)P'_{-\frac{1}{2}}(\mu_0)P_{s-\frac{1}{2}}(-\mu_0)+Bs^{-2}P'_{s-\frac{1}{2}}(-\mu_0)\operatorname{cosec}^2\alpha$$

$$+\frac{B}{2s}P'_{s-\frac{1}{2}}(-\mu_0)\int_{-\infty}^{\infty}\frac{P'_{iy-\frac{1}{2}}(-\mu_0)P_{iy-\frac{1}{2}}(\mu_0)}{y^2+s^2}\operatorname{sech}\pi y\,dy$$

$$\sim\left[\left(\frac{A}{2}\operatorname{cosec}^2\alpha\right)s^{-1}+O(s^{-2})\right]P_{s-\frac{1}{2}}(-\mu_0)$$

$$+\left(\frac{B}{2}\operatorname{cosec}^2\alpha\right)s^{-2}P_{s-\frac{1}{2}}(-\mu_0)\tag{A.22}$$

the final approximation for large s being obtained from that given for the infinite integral at the final step of (A.18).

Lastly, as regards the line integral terms in (A.16), we have first by the formulas (A.17)–(A.20)

$$\sum\frac{P_m(-\mu_0)}{(m+\frac{1}{2})^2-z^2}[A(m+\tfrac{1}{2})P_m(-\mu_0)+BP_m'(-\mu_0)]$$

$$=A\frac{\pi}{2}P_{z-\frac{1}{2}}(-\mu_0)P_{z-\frac{1}{2}}(\mu_0)\sec\pi z$$

$$+\frac{1}{2}B\left\{\frac{\pi}{z}P'_{z-\frac{1}{2}}(-\mu_0)P_{z-\frac{1}{2}}(\mu_0)\sec\pi z\right.$$

$$\left.-\int_{-\infty}^{\infty}\frac{P'_{iy-\frac{1}{2}}(-\mu_0)P_{iy-\frac{1}{2}}(\mu_0)}{y^2+z^2}\operatorname{sech}\pi y\,dy\right\}$$

$$\sim\pm\frac{Ai}{2\sin\alpha}z^{-1}+BO(z^{-2})\tag{A.23}$$

when $|z|$, $|\operatorname{Im}(z)|$ are large compared with unity (>3 will do). This last step follows from the approximation for the integral given at the final step of (A.18) and the result for the coefficient of A from

$$P_{z-\frac{1}{2}}(-\mu_0)P_{z-\frac{1}{2}}(\mu_0)\sec\pi z\sim\pm\left(\frac{i}{\sin\alpha}\right)z^{-1}\tag{A.24}$$

true when $|z|$, $|\operatorname{Im}(z)|\gg 1$, the $\pm$ being taken according as $\operatorname{Im}(z)\gtrless 0$.

When s is large we may approximate to the functions of z in the integrand of the line integral under consideration by their asymptotic forms given above when z is large—for this (from consideration of the integrand) clearly involves a relative error $O(s^{-2})$—which is negligible for large s compared with the dominant term of the result (A.25) below.

Doing this, and using (A.23) above together with (A.3) and (A.4), we obtain

$$\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{1}{z(s^2-z^2)} \frac{\psi'_{z-\frac{1}{2}}}{\psi_{z-\frac{1}{2}}} \frac{P'_{z-\frac{1}{2}}(-\mu_0)}{P_{z-\frac{1}{2}}(-\mu_0)} \sum_0^\infty \frac{P_m(-\mu_0)}{(m+\frac{1}{2})^2 - z^2}$$
$$\times [A(m+\tfrac{1}{2})P_m(-\mu_0) + BP_m{}'(-\mu_0)]\, dz$$
$$\sim 2\int_0^{\kappa+i\infty} \frac{1}{z(s^2-z^2)} \frac{z}{x} \frac{iz}{\sin\alpha} \left(\frac{Ai}{2\sin\alpha}\right) z^{-1}\, dz + O(s^{-2})$$
$$\sim -\frac{\pi A i}{2x\sin^2\alpha} s^{-1} + O(s^{-2}) \tag{A.25}$$

Thus, finally, (A.16) gives, for large enough s

$$(A\operatorname{cosec}^2\alpha)s^{-1}P_{s-\frac{1}{2}}(-\mu_0) + \left(B\operatorname{cosec}^2\alpha - \frac{iK}{2}\right)P'_{s-\frac{1}{2}}(-\mu_0)$$
$$= \left(\frac{A}{2}\operatorname{cosec}^2\alpha\right)s^{-1}P_{s-\frac{1}{2}}(-\mu_0) + \left(\frac{A}{2}\operatorname{cosec}^2\alpha\right)s^{-1}P_{s-\frac{1}{2}}(-\mu_0)$$
$$+ \left(\frac{B}{2}\operatorname{cosec}^2\alpha\right)s^{-2}P'_{s-\frac{1}{2}}(-\mu_0) \tag{A.26}$$

Hence

$$B = ik\sin^2\alpha \tag{A.27}$$

But this method (it has transpired) proved insufficient to determine the constant A, for the above equation is satisfied for *all* values of A.

Actually, A can be determined by a detailed consideration of the effect inclusion of its contribution $As^{-1}P_{s-\frac{1}{2}}(-\cos\alpha)$ in the expression for $C_{s-\frac{1}{2}}$ would have upon the rim singularity.

Inclusion of this extra term in text expression (10) for $C_{s-\frac{1}{2}}$ will produce an extra term proportional to $n^{-2}(l/r)^n\cos[(n+\frac{1}{2})\alpha - \frac{1}{4}\pi]\cos[(n+\frac{1}{2})\theta - \frac{1}{4}\pi]$ on the right of text expression (14) for Π_n, and so an extra series term proportional to $\sum n^{-1}(l/r)^n\cos[(n+\frac{1}{2})\alpha - \frac{1}{4}\pi]\sin[(n+\frac{1}{2})\theta - \frac{1}{4}\pi]$ on the right of text expression (16) for H_ϕ.

However, as the rim is approached, we have

$$2\sum n^{-1}(l/r)^n\cos[(n+\tfrac{1}{2})\alpha - \tfrac{1}{4}\pi]\sin[(n+\tfrac{1}{2})\theta - \tfrac{1}{4}\pi]$$
$$= -\sum n^{-1}(l/r)^n\{\sin[(n+\tfrac{1}{2})(\theta-\alpha)] - \cos[(n+\tfrac{1}{2})(\theta+\alpha)]\}$$
$$\sim \operatorname{Im}\left\{e^{\frac{1}{2}i(\theta-\alpha)}\ln\left[1 - \frac{l}{r}e^{i(\theta-\alpha)}\right] - \operatorname{Re}\left\{e^{\frac{1}{2}i(\theta-\alpha)}\ln\left[1 - \frac{l}{r}e^{i(\theta+\alpha)}\right]\right\}\right.$$
$$\sim -\frac{1}{2}\frac{\epsilon}{l}\sin\gamma\cdot\ln\frac{l}{\epsilon} - \gamma - \ln(2\sin\alpha) \tag{A.28}$$

and so the addition of the A-term does not affect the $\mathbf{H}$ singularity.

To examine the effect upon the **E** singularity we merely have to study the above addition to H_ϕ in the formulas

$$E_r \sim \frac{1}{ikl}\left(\frac{\partial H_\phi}{\partial \theta}\right)$$
$$E_\theta \sim -\frac{1}{ik}\left(\frac{\partial H_\phi}{\partial r}\right) \tag{A.29}$$

Remembering that, in the above

$$r = l + \epsilon \cos\gamma$$
$$l(\theta - \alpha) = \epsilon \sin\gamma \tag{A.30}$$

and performing the indicated differentiations, we find that the effect of the A-term is to add to the text expressions, for E_r a term proportional to $\epsilon^{-1} \cos\gamma$ and for E_θ a term proportional to $\epsilon^{-1}\sin\gamma$.

With regard to the effect of the addition of this A-term in the expression for large order $C_{s-\frac{1}{2}}$ upon the region I formulas, following the text argument down from the beginning of Section III, B, the first point where the difference makes itself felt is at Eq. (33), where an extra term

$$-klA \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})P_n'(\mu\ )P_n(\mu\ )}{(n+\frac{1}{2})^2 - (\beta_j + \frac{1}{2})^2} \tag{A.31}$$

is introduced on the right hand side.

But, by contour integration of $sP'_{s-\frac{1}{2}}(\mu_0)P_{s-\frac{1}{2}}(-\mu_0) \sec s\pi/[s^2 - (\beta_j + \frac{1}{2})^2]$ around the usual large semicircle, bounded on the left by the imaginary axis, the sum is equal to

$$-\frac{1}{2}\int_{-\infty i}^{\infty i} \frac{sP'_{s-\frac{1}{2}}(\mu_0)P_{s-\frac{1}{2}}(-\mu_0)}{s^2 - (\beta_j + \frac{1}{2})^2} \sec s\pi \, ds$$

since there is no residue of

$$sP'_{s-\frac{1}{2}}(\mu_0)P_{s-\frac{1}{2}}(-\mu_0) \sec s\pi/[s^2 - (\beta_j + \tfrac{1}{2})^2] \qquad \text{at} \qquad s = \beta_j + \tfrac{1}{2}$$

But the above integral is zero, for the integrand is an odd function of s.

Hence the addition of the A-term to formula (10) of the text for the $C_{s-\frac{1}{2}}$ *does not affect the region I formulas for the approach to the rim.*

Hence, by considering the requirement that E_θ be continuous across the region I-II boundary $\gamma = \pi/2$ for a path around, but near to, the rim, we conclude from (20) that the additive term (proportional to $\epsilon^{-1}\sin\gamma$, as above noted) has to be zero at $\gamma = \pi/2$. Hence $A = 0$.

Appendix II. Two Legendre Function Theory Results

By the theory of Legendre functions, we have

$$P_n(-\mu) = \cos n\pi P_n(\mu) - (2 \sin n\pi/\pi)Q_n(\mu)$$

Also

$$P_n(\mu)Q_n'(\mu) - P_n'(\mu)Q_n(\mu) = 1/(1 - \mu^2)$$

Hence,

$$P_n(\mu)P_n'(-\mu) + P_n'(\mu)P_n(-\mu) = \frac{2}{\pi}\frac{\sin n\pi}{1 - \mu^2} \tag{A2.1}$$

Also, substituting $n = \beta_j$, $\mu = \mu_0$ in this we find

$$P_{\beta_j}(\mu_0)P'_{\beta_j}(-\mu_0) = \frac{2}{\pi}\frac{\sin \pi\beta_j}{1 - {\mu_0}^2} \tag{A2.2}$$